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On Birkhoff's theorem in a general class of scalar–tensor theories

S B Dutta Choudhury† and D Bhattacharya‡

† Department of Physics, St Anthony's College, Shillong 793003, India
‡ Department of Physics, Jadavpur University, Calcutta 700032, India

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Abstract. For a general class of scalar–tensor theories of gravitation proposed by Nordtvedt, it is shown that Birkhoff's theorem holds both in vacuum as well as in the presence of an electromagnetic field when the scalar field is time independent.

1. Introduction

In general relativity every spherically symmetric vacuum field of gravitation must be static. This fact is known as Birkhoff's (1923) theorem. The possibility of the theorem being valid in other theories of gravitation have been examined by several authors (Schücking 1959, O'Hanlon and Tupper 1972, Reddy 1973, Krori and Nandy 1977). They showed that in different forms of scalar–tensor theories, e.g. Jordan (1959), Brans and Dicke (1961), Ross (1972), Sen and Dunn (1971), Birkhoff's theorem holds when the scalar field is time independent.

Again, in general relativity, every spherically symmetric electromagnetic field must be static (Hoffmann 1932, Das 1960). This so called generalisation of Birkhoff's theorem was found to be valid in the scalar–tensor theory of Sen and Dunn (1971) (Reddy 1977) when the scalar field is time independent.

In this paper we have established that the time invariance of the scalar field is a sufficient condition for Birkhoff's theorem to hold in the case of the most general class of scalar–tensor theory proposed by Nordtvedt (1970), both in vacuum as well as in the presence of an electromagnetic field. Further, the proof is generalised for all fields with a particular structure of the energy–momentum tensor. It can be shown that the Nordtvedt class includes as special cases the theories of Jordan (1959), Brans and Dicke (1961) and Barkar (1978).

2. Birkhoff's theorem in Nordtvedt's theory of gravitation

We start with the case where the scalar field is coupled to an electromagnetic field for which the Nordtvedt–Maxwell field equations are

$$\Box\psi = -\frac{(\mathrm{d}\omega/\mathrm{d}\psi)}{(2\omega+3)}\psi_{,\alpha}\psi^{,\alpha} \tag{1}$$

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$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=\frac{2}{\psi}(F^{\beta}_{\mu}F_{\nu\beta}-\tfrac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta})$$

$$-\frac{\omega}{\psi^2}(\psi_{,\mu}\psi_{,\nu}-\tfrac{1}{2}g_{\mu\nu}\psi_{,\alpha}\psi^{,\alpha})-\frac{1}{\psi}(\psi_{\mu;\nu}-g_{\mu\nu}\Box\psi) \tag{2}$$

$$F^{\mu\nu}_{;\nu}=0 \tag{3}$$

$$^*F^{\mu\nu}_{;\nu}=0 \tag{4}$$

where the function $\omega(\psi)$ is an arbitrary (positive definite) function of the scalar field ψ and $F^{\mu\nu}$ is the contravariant antisymmetric electromagnetic field tensor with $^*F^{\mu\nu}$ as its tensor dual. Special choices for $\omega(\psi)$ yield the theories of Jordan (1959), Brans and Dicke (1961) and Barkar (1978).

Let the space–time be described by the spherically symmetric line element of the form

$$ds^2=e^{\nu}\,dt^2-e^{\lambda}\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2) \tag{5}$$

where λ and ν are functions of both r and t. Again, by virtue of spherical symmetry, the surviving components of $F^{\mu\nu}$ are

$$F_{14}=-F_{41}\qquad F_{23}=-F_{32}.$$

Also, in view of the metric (5), it follows from equation (3) that

$$F_{14}=\left(\frac{q}{r^2}\right)e^{(\lambda+\nu)/2} \tag{6}$$

$$F_{23}=m\sin\theta \tag{7}$$

where q and m are arbitrary constants and can be interpreted, respectively, as the charge and the magnetic pole strength of the point source. Now, the Nordtvedt–Maxwell field equations (1) and (2) for the metric (5) reduce to

$$-e^{-\lambda}\left(\frac{\nu'}{r}+\frac{1}{r^2}\right)+\frac{1}{r^2}$$

$$=\frac{1}{\psi}\frac{q^2+m^2}{r^4}-e^{-\lambda}\left(\frac{\psi''}{\psi}-\frac{\lambda'\psi'}{2\psi}+\frac{\omega\psi'^2}{2\psi^2}+\frac{(d\omega/d\psi)\psi'^2}{\psi(2\omega+3)}\right)$$

$$+e^{-\nu}\left(\frac{\dot\lambda\dot\psi}{2\psi}-\frac{\omega\dot\psi^2}{2\psi^2}+\frac{(d\omega/d\psi)\dot\psi^2}{\psi(2\omega+3)}\right)-e^{-\lambda}\left(\frac{\nu''}{2}-\frac{\lambda'\nu'}{4}-\frac{\nu'^2}{4}+\frac{\nu'-\lambda'}{2r}\right)+e^{-\nu}\left(\frac{\ddot\lambda}{2}+\frac{\dot\lambda^2}{4}-\frac{\dot\lambda\dot\nu}{4}\right) \tag{8}$$

$$=\frac{1}{\psi}\frac{q^2+m^2}{r^4}+e^{-\lambda}\left(\frac{\omega\psi'^2}{2\psi^2}-\frac{\psi'}{r\psi}-\frac{(d\omega/d\psi)\psi'^2}{\psi(2\omega+3)}\right)-e^{-\nu}\left(\frac{\omega\dot\psi^2}{2\psi^2}-\frac{(d\omega/d\psi)\dot\psi^2}{\psi(2\omega+3)}\right) \tag{9}$$

$$e^{-\lambda}\left(\frac{\lambda'}{r^2}-\frac{1}{r^2}\right)+\frac{1}{r^2}$$

$$=\frac{1}{\psi}\frac{q^2+m^2}{r^4}+e^{-\lambda}\left(\frac{\omega\psi'^2}{2\psi^2}-\frac{\nu'\psi'}{2\psi}-\frac{(d\omega/d\psi)\psi'^2}{\psi(2\omega+3)}\right)$$

$$+e^{-\nu}\left(\frac{\omega\dot\psi^2}{2\psi^2}+\frac{\ddot\psi}{\psi}-\frac{\dot\lambda\dot\psi}{2\psi}+\frac{(d\omega/d\psi)\dot\psi^2}{\psi(2\omega+3)}\right) \tag{10}$$

$$\frac{\dot{\lambda}}{r}=\frac{\omega\psi'\dot{\psi}}{\psi^2}-\frac{\dot{\psi}'}{\psi}-\frac{\dot{\lambda}\psi'}{2\psi}-\frac{\nu'\dot{\psi}}{2\psi} \tag{11}$$

$$-\mathrm{e}^{-\lambda}\left[\psi''+\psi'\left(\frac{2}{r}+\frac{\nu'-\lambda'}{2}\right)\right]+\mathrm{e}^{-\nu}[\ddot{\psi}-\tfrac{1}{2}(\dot{\nu}-\dot{\lambda})\dot{\psi}]$$
$$=\frac{\mathrm{e}^{-\lambda}(\mathrm{d}\omega/\mathrm{d}\psi)\psi'^2}{(2\omega+3)}-\frac{\mathrm{e}^{-\nu}(\mathrm{d}\omega/\mathrm{d}\psi)\dot{\psi}^2}{(2\omega+3)} \tag{12}$$

where a prime denotes partial differentiation with respect to r and a dot that with respect to t.

When the scalar field is time independent, i.e. when $\dot{\psi}=0$, we get from (11)

$$\dot{\lambda}\left(1+\frac{r\psi'}{2\psi}\right)=0, \tag{13}$$

which gives either

$$\dot{\lambda}=0 \tag{14}$$

or

$$\frac{r\psi'}{2\psi}=-1 \qquad \text{i.e. } \psi=\frac{\psi_0}{r^2} \tag{15}$$

where ψ_0 is a constant of integration. Now, subtracting (8) from (10), we get, with $\dot{\psi}=0$,

$$\frac{(\lambda'+\nu')}{r}\left(1+\frac{r\psi'}{2\psi}\right)=\left(\frac{\psi''}{\psi}+\frac{\omega\psi'^2}{\psi^2}\right). \tag{16}$$

With (15) as the correct solution, we get from (16)

$$\left(\frac{\psi''}{\psi}+\frac{\omega\psi'^2}{\psi^2}\right)=0$$

and this gives for $\psi=\psi_0/r^2$

$$2(2\omega+3)/r^2=0. \tag{17}$$

But in Nordtvedt's theory $(2\omega+3)\neq 0$, so the only other possibility is valid, i.e. $\dot{\lambda}=0$.

Again, differentiating (16) with respect to t one obtains

$$\dot{\nu}'=0. \tag{18}$$

Hence we get

$$\nu=R(r)+T(t) \tag{19}$$

where $R(r)$ and $T(t)$ are arbitrary functions of r and t respectively. Introducing a purely temporal transformation $\mathrm{d}\bar{t}=\mathrm{e}^{T/2}\,\mathrm{d}t$ and then denoting the time coordinate $\bar{t}$ again by t, it follows, in view of (17), that the metric (5) is static.

For the spherically symmetric purely vacuum gravitational field, the proof readily follows from above simply by putting $q=m=0$.

3. Discussion

(*a*) In the presence of the cosmological constant $\Lambda \neq 0$, with any arbitrary energy–momentum tensor T^μ_ν, the relevant equations with $\dot{\psi} = 0$ are

$$-G^0_0 = \mathrm{e}^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi}{\psi} T^0_0$$

$$+\mathrm{e}^{-\lambda}\left(\frac{\omega\psi'^2}{2\psi^2} - \frac{\nu'\psi'}{2\psi} - \frac{(\mathrm{d}\omega/\mathrm{d}\psi)\psi'^2}{\psi(2\omega+3)}\right) + \Lambda\delta^0_0 \tag{20}$$

$$-G^1_1 = -\mathrm{e}^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} = \frac{8\pi}{\psi} T^1_1$$

$$-\mathrm{e}^{-\lambda}\left(\frac{\psi''}{\psi} - \frac{\lambda'\psi'}{2\psi} + \frac{\omega\psi'^2}{2\psi^2} + \frac{(\mathrm{d}\omega/\mathrm{d}\psi)\psi'^2}{\psi(2\omega+3)}\right) + \Lambda\delta^1_1 \tag{21}$$

$$G^1_4 = \frac{\dot{\lambda}}{r} = \frac{8\pi}{\psi} T^1_4 - \frac{\dot{\lambda}\psi'}{2\psi}. \tag{22}$$

Proceeding as above, we can arrive at the result provided $T^1_1 = T^0_0$ and $T^1_4 = 0$. As $T^2_2 = T^3_3$, by virtue of spherical symmetry, it is apparent that the proof is solely dependent on the structure of the energy–momentum tensor, i.e. the proof is valid whenever T^μ_ν has the structure $T^1_1 = T^0_0$, $T^2_2 = T^3_3$ and $T^\mu_\nu = 0$ $(\mu \neq \nu)$ or, in other words, when T^μ_ν has two mutually orthogonal invariant subspaces of dimension two, each composed entirely of eigenvectors. As such, it is also evident that the proof does not really depend on Maxwell's equations (3) and (4), but only on the algebraic structure of T^μ_ν. (T^μ_ν belongs to a special subclass of Segré type [(11)(11)].)

(*b*) The proof as given above relies on the existence of the curvature coordinates. For a more general line element

$$\mathrm{d}s^2 = \mathrm{e}^{\nu(r,t)}\,\mathrm{d}t^2 - \mathrm{e}^{\lambda(r,t)}\,\mathrm{d}r^2 - \mathrm{e}^{\mu(r,t)}\,\mathrm{d}\Omega^2$$

the proof is not so apparent.

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